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Matrix-valued continued fractions ☆

Huan-xi Zhao[a,*] and Gongqin Zhu[b]

[a] Department of Mathematics, University of Science & Technology of China, Anhui Hefei 230026, People's Republic of China

[b] Department of Applied Mathematics, Central-South University, Changsha 410083, People's Republic of China

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Abstract

We discuss the properties of matrix-valued continued fractions based on Samelson inverse. We begin to establish a recurrence relation for the approximants of matrix-valued continued fractions. Using this recurrence relation, we obtain a formula for the difference between mth and nth approximants of matrix-valued continued fractions. Based on this formula, we give some necessary and sufficient conditions for the convergence of matrix-valued continued fractions, and at the same time, we give the estimate of the rate of convergence. This paper shows that some famous results in the scalar case can be generalized to the matrix case, even some of them are exact generalizations of the scalar results.

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1. Introduction

A continued fraction is an expression of the form

$$b_0 + \frac{a_1|}{|\, b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n} + \cdots, \tag{1.1}$$

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*Corresponding author.
E-mail address: hxzhao@csu.edu.cn (H.-xi Zhao).

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where the a_i and b_i are real (or complex) numbers or functions, and the theories and properties of continued fractions are well known in the scalar case. But more general forms of continued fraction (1.1) where a_i and b_i are no longer real or complex numbers are possible. They may be vectors, matrixes or the elements of some Banach algebra. There clearly exists some interest in extending these kinds of theories and properties to the vector or matrix case, and these extensions occur in computations of various mathematical, physical and control problems [see [2,5,7–19,23,25,30,31]]. For example, in control theory for expansion of the transfer function of multivariate control systems, in theoretical physics for investigations of the Brownian motion and of the an harmonic oscillator eigenvalues, in perturbation theory, as well as in rational interpolation and approximation. Here, we especially refer to the works of Wynn et al. In 1963, Wynn used continued fractions and generalized inverses for the reciprocals of vector-valued quantities, and proposed a method of rational interpolation of vector-valued quantities given on a set of distinct interpolation points [see [32,33]]. In [7–13,25,30], Graves-Morris, etc., showed that the generalized (Samelson) inverse of vector can be used to define vector-valued Thiele type rational interpolations and vector Padé approximants, and indicated that generalized inverse vector rational interpolations had wide applications in the modal analysis of vibrating structures and the solution of integral equations. In [14,15,17,18,34,35], Zhu Gong-qin and C.Q. Gu, etc., introduced the generalized inverse vector continued fraction approximation of vector-valued functions and indicated that the generalized inverse vector rational interpolant can be extended to the bivariate and the matrix case. In [35], by defining a kind of transformation from matrix to vector, some results on vector-valued continued fraction can be transferred to those corresponding to matrix-valued continued fraction, but, as we know, vector-valued continued fraction is obviously special case of matrix-valued continued fraction. Therefore, in this paper, we restrict our considerations to the matrix case. Here, we should point out that a different matrix inverse can give the different definition of a matrix continued fraction, so we can extend definition from the scalar case to the matrix case in the following three ways. In [1,2,19,23,24,26], the fraction of matrix is AB^{-1} with the classical inverse matrix, and in [3,6,28,29,31], the inverse of matrix is a partial inverse, namely, if the matrix $B=(b_{ij})_{m\times n}$, then the partial inverse of matrix B is

$$\frac{1}{B}=(B)_P^{-1}=B_2B_1^{-1}$$

here,

$$B_1=\begin{pmatrix} b_{m,1} & \cdots & \cdots & b_{m,n} \\ 1 & \cdots & \cdots & 0 \\ 0 & \ddots & & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix}_{n\times n},$$

$$B_2 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ b_{11} & \cdots & \cdots & b_{1,n} \\ & \cdots & \cdots & \\ b_{m-1,1} & & & b_{m-1,n} \end{pmatrix}_{m\times n}.$$

Therefore, for convenience the above matrix continued fractions defined by the classical matrix inverse or partial matrix inverse are called the classical inverse matrix continued fractions or the partial inverse matrix continued fractions, respectively, and they are both examples of noncommutative continued fractions in Banach spaces. In our case, the evaluation process is based on the use of the generalized inverse for matrix [see Section 2 or [14,16,35]], so the considered continued fractions which can be named matrix-valued continued fractions or generalized inverse matrix continued fractions in this paper, is not the same as these in [1–3,5,19,23,24,26–28,31]. In [16], it is shown that the generalized inverse is efficient in matrix continued fraction interpolation problems as compared with classical matrix inverse, and as compared to the existing matrix Padé approximants, the generalized inverse matrix Padé approximation has a lot of advantages. For example, first, it does not need multiplication of matrices in the construction process, hence, we do not have to define left- and right-handed approximants, and it may be useful in the noncommutativity problems of the matrix multiplication. Second, the generalized inverse matrix Padé approximation can be applied to singular or rectangle matrices.

For the classical inverse or partial inverse matrix continued fractions, some properties about them are obtained [see [1,3,5,24,27,28,31]], But in our case, not much are known, and many problems are not still answered. For example, can the classical three term recurrence relation be generalized in a practical way to our case? How to get the properties of the approximants of MVCF? In particular, the applications of continued fractions are often tied to their possible convergence, therefore, the convergence criteria are important in the theory of matrix-valued continued fractions. But up to now, the convergence property for MVCF has not been reported. In scalar case, the main methods we use to derive the convergence criteria for continued fractions are based upon three term recurrence relation or some very nice mapping properties of linear fractional transformations or value regions techniques for continued fractions. However, in more general case, such as in our case, it should be noted that it appear difficult to find recurrence relations similar to the famous three-term recurrence relation which can be used to derive surprisingly good results in the convergence theory for continued fractions. Therefore, those methods cannot be used in the proofs of the convergence criteria for MVCF. In [34], using a particular technique, a simple Pringsheim convergence theorem was proven for vector-valued continued fraction of the form $K[1/\vec{b}_k]$, but it may be difficult or even impossible to prove best known and(or) the widest applicable classical convergence theorems for MVCF directly using the method introduced in [3,5,11,20–22,24,27,28,31,34]. In this paper, we answer some of these questions. We firstly establish a recurrence relation for MVCF, and using this recurrence, we construct a formula for the difference between mth and nth approximants of MVCF. Based on this formula, we give a approach to prove the convergence properties of MVCF. This paper shows that some famous results in the scalar case are similar to those in the matrix case, even some of them are exact generalizations or improvement of the scalar results.

2. Recurrence relation

By a matrix-valued continued fraction, we mean an expression of the form

$$B_0 + \sum_{i=1}^{\infty} \frac{a_i|}{|B_i}, \tag{2.1}$$

where $B_i \in \mathbb{C}^{k\times l},\ a_i \in \mathbb{C}$, for $i = 0, 1, 2, \ldots$.

The above evaluation process is based on the use of the generalized (Samelson) inverse for matrix:

$$A^{-1} = \bar{A}/||A||^2, \quad A \neq 0, \tag{2.2}$$

where $\bar{A}$ denotes the complex conjugate of matrix A and

$$||A||^2 = tr(A(\bar{A})^T) = \sum_{i=1}^{k} \sum_{j=1}^{l} |a_{ij}|^2, \quad A = (a_{ij}) \in \mathbb{C}^{k\times l},\ k, l \in \mathbb{N}$$

where $(\bar{A})^T$ denotes the complex conjugate transpose of matrix A and $|a_{ij}|$ is the modulus of a_{ij}. Making use of the above generalized inverse for matrix, matrix-valued continued fractions have been discussed in [14,17,18].

Similar to the scalar case, the nth approximant of MVCF is defined as

$$R_n = B_0 + \frac{a_1|}{|B_1} + \cdots + \frac{a_n|}{|B_n} \quad (n = 0, 1, \ldots). \tag{2.3a}$$

Clearly, R_n is a rational expression

$$R_n = \frac{P_n}{Q_n},$$

where P_n and Q_n are, respectively, called the nth numerator and denominator of (2.3a), and they are actually defined later as P_n^0, Q_n^0 via (2.4)–(2.6).

The truncation

$$R_n^k = B_k + \frac{a_{k+1}|}{|B_{k+1}} + \cdots + \frac{a_n|}{|B_n} \quad (n = 0, 1, \ldots.;\ k = 0, 1, \ldots, n) \tag{2.3b}$$

is called the kth tail of the nth approximant of MVCF (2.1).

Similarly, R_n^k is a rational expression

$$R_n^k = \frac{P_n^k}{Q_n^k},$$

where P_n^k and Q_n^k, are, respectively, called the kth numerator and denominator of (2.3b).

Clearly, we have

$$R_n = \frac{P_n}{Q_n} = R_n^0 = \frac{P_n^0}{Q_n^0}.$$

Now, we want to establish a recurrence relation for P_n^k and Q_n^k. Namely, we have

Theorem 1. *For any positive integer n, let*

$$P_n^n = B_n, \;\; Q_n^n = 1, \;\; Q_n^{n-1} = ||B_n||^2 \quad (n = 1, 2, \dots), \tag{2.4}$$

$$P_n^i = B_i Q_n^i + a_{i+1} \bar{P}_n^{i+1} \quad (i = n-1, \dots, 0), \tag{2.5}$$

$$Q_n^i = ||B_{i+1}||^2 Q_n^{i+1} + 2\,\mathrm{Re}[tr(\bar{a}_{i+2} P_n^{i+2} B_{i+1}^T)] + |a_{i+2}|^2 Q_n^{i+2} \quad (i = n-2, \dots, 1, 0), \tag{2.6}$$

then we have

$$(1)\;\; Q_n^i \geqslant 0 \quad (n = 0, 1, \dots, \;\; i = n, n-1, \dots, 1, 0), \tag{2.7}$$

$$(2)\;\; ||P_n^i||^2 = Q_n^i Q_n^{i-1} \quad (i = n, n-1, \dots, 1), \tag{2.8}$$

$$(3)\;\; R_n^i = \frac{P_n^i}{Q_n^i} = B_i + \frac{a_{i+1}|}{|B_{i+1}} + \frac{a_{i+2}|}{|B_{i+2}} + \cdots + \frac{a_n|}{|B_n} \quad (i = n, n-1, \dots, 1, 0). \tag{2.9}$$

Proof. The proof is performed by induction.

(1) For any $k \leqslant n$, from (2.6), we have

$$\begin{aligned} Q_n^k &= ||B_{k+1}||^2 Q_n^{k+1} + 2\mathrm{Re}[tr(\bar{a}_{k+2} P_n^{k+2} B_{k+1}^T)] + |a_{k+2}|^2 Q_n^{k+2} \\ &\geqslant ||B_{k+1}||^2 Q_n^{k+1} - |tr(\bar{a}_{k+2} P_n^{k+2} B_{k+1}^T) + tr(a_{k+2} (\bar{B}_{k+1})^T \bar{P}_n^{k+2})| + |a_{k+2}|^2 Q_n^{k+2} \\ &\geqslant ||B_{k+1}||^2 Q_n^{k+1} - 2|a_{k+2}| \cdot ||P_n^{k+2}|| \cdot ||B_{k+1}|| + |a_{k+2}|^2 Q_n^{k+2} \\ &= (||B_{k+1}|| \sqrt{Q_n^{k+1}} - |a_{k+2}| \sqrt{Q_n^{k+2}})^2 \geqslant 0 \end{aligned}$$

which completes the proof of relation (2.7).

(2) For $i = n$, equality (2.8) is immediate from relation (2.4).

Now, let us assume that for $i = k, 1 \leqslant k < n$, equality (2.8) is true. We shall prove it for $i = k - 1$. It follows from (2.5), (2.6) and by the induction hypothesis that

$$\begin{aligned}||P_n^{k-1}||^2 &= tr\{(B_{k-1}Q_n^{k-1} + a_k\bar{P}_n^k)((\bar{B}_{k-1})^T Q_n^{k-1} + \bar{a}_k(P_n^k)^T)\} \\ &= ||B_{k-1}||^2(Q_n^{k-1})^2 + Q_n^{k-1}\{\overline{[tr(\bar{a}_k P_n^k B_{k-1}^T)]} \\ &\quad + tr[\bar{a}_k P_n^k B_{k-1}^T]\} + |a_k|^2||P_n^k||^2 \\ &= ||B_{k-1}||^2(Q_n^{k-1})^2 + Q_n^{k-1}2\mathrm{Re}\{tr(\bar{a}_k P_n^k B_{k-1}^T)\} + |a_k|^2||P_n^k||^2 \\ &= Q_n^{k-1}\{||B_{k-1}||^2 Q_n^{k-1} + 2\mathrm{Re}\{tr(\bar{a}_k P_n^k B_{k-1}^T)\} + |a_k|^2 Q_n^k\} \\ &= Q_n^{k-1}Q_n^{k-2}\end{aligned}$$

which implies equality (2.8) is valid.

(3) When $i = n$, from (2.4), formula (2.9) is obviously true.

Next, let equality (2.9) hold for $i = k$. Then when $i = k-1$, from (2.5), (2.8) and by the induction hypothesis, we have

$$\begin{aligned}\frac{P_n^{k-1}}{Q_n^{k-1}} &= \frac{B_{k-1}Q_n^{k-1} + a_k\bar{P}_n^k}{Q_n^{k-1}} = B_{k-1} + \frac{a_k\bar{P}_n^k}{Q_n^{k-1}} = B_{k-1} + \frac{a_k||P_n^k||^2}{Q_n^{k-1}P_n^k} \\ &= B_{k-1} + \frac{a_k}{\frac{P_n^k}{Q_n^k}} = B_{k-1} + \frac{a_k|}{|B_k} + \frac{a_{k+1}|}{|B_{k+1}} + \cdots + \frac{a_n|}{|B_n}\end{aligned}$$

which completes the proof of (2.9) □

3. Approximants formulae

In this section, we give a formula for the difference between mth and nth approximants of matrix-valued continued fractions. In general, those formulae are important for truncation error estimate and convergence theory of MVCF.

Theorem 2. *The formula*

$$\left|\left|\frac{P_{n+m}}{Q_{n+m}} - \frac{P_n}{Q_n}\right|\right| = \left|\left|\frac{P_{n+m}^0}{Q_{n+m}^0} - \frac{P_n^0}{Q_n^0}\right|\right| = \frac{|a_1|\cdots|a_{n+1}|\sqrt{Q_{n+m}^{n+1}}}{\sqrt{Q_n^0}\sqrt{Q_{n+m}^0}} \quad \text{for any } n, m \in \mathbb{N} \tag{3.1}$$

holds true for two convergents P_{n+m}/Q_{n+m} and P_n/Q_n of matrix-valued continued fraction (2.1).

Proof. Let

$$\left|\left|\frac{P_{n+m}^k}{Q_{n+m}^k} - \frac{P_n^k}{Q_n^k}\right|\right| = \frac{\Delta_k}{Q_n^k Q_{n+m}^k}, \tag{3.2}$$

where

$$\Delta_k = ||Q_n^k P_{n+m}^k - Q_{n+m}^k P_n^k||.$$

Using relation (2.5), we get

$$\begin{aligned}\Delta_k &= ||(B_k Q_{n+m}^k + a_{k+1}\bar{P}_{n+m}^{k+1})Q_n^k - (B_k Q_n^k + a_{k+1}\bar{P}_n^{k+1})Q_{n+m}^k||\\ &= ||a_{k+1}(\bar{P}_{n+m}^{k+1}Q_n^k - \bar{P}_n^{k+1}Q_{n+m}^k)|| = ||\bar{a}_{k+1}(P_{n+m}^{k+1}Q_n^k - P_n^{k+1}Q_{n+m}^k)||.\end{aligned}$$

From this it follows that

$$\begin{aligned}\Delta_k^2 &= tr\{\bar{a}_{k+1}(P_{n+m}^{k+1}Q_n^k - P_n^{k+1}Q_{n+m}^k)a_{k+1}[(\bar{P}_{n+m}^{k+1})^T Q_n^k - (\bar{P}_n^{k+1})^T Q_{n+m}^k]\}\\ &= |a_{k+1}|^2(||P_{n+m}^{k+1}||^2(Q_n^k)^2\\ &\quad - 2\,\mathrm{Re}[tr(P_{n+m}^{k+1}(\bar{P}_n^{k+1})^T)]Q_n^k Q_{n+m}^k + ||P_n^{k+1}||^2(Q_{n+m}^k)^2)\\ &= |a_{k+1}|^2||P_{n+m}^{k+1}||^2||P_n^{k+1}||^2\left(\frac{||P_n^{k+1}||^2}{(Q_n^{k+1})^2} - \frac{2\,\mathrm{Re}[tr(P_{n+m}^{k+1}(\bar{P}_n^{k+1})^T)]}{Q_n^{k+1}Q_{n+m}^{k+1}} + \frac{||P_{n+m}^{k+1}||^2}{(Q_{n+m}^{k+1})^2}\right)\\ &= |a_{k+1}|^2||P_{n+m}^{k+1}||^2||P_n^{k+1}||^2||P_n^{k+1}/Q_n^{k+1} - P_{n+m}^{k+1}/Q_{n+m}^{k+1}||^2.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\Delta_k &= |a_{k+1}|||P_{n+m}^{k+1}||||P_n^{k+1}||||P_n^{k+1}/Q_n^{k+1} - P_{n+m}^{k+1}/Q_{n+m}^{k+1}||\\ &= \frac{|a_{k+1}|||P_n^{k+1}||||P_{n+m}^{k+1}||}{Q_n^{k+1}Q_{n+m}^{k+1}}\Delta_{k+1}.\end{aligned}$$

Continuing the above process, it follows that

$$\begin{aligned}\Delta_k &= \frac{|a_{k+1}|||P_n^{k+1}||||P_{n+m}^{k+1}||}{Q_n^{k+1}Q_{n+m}^{k+1}}\Delta_{k+1}\\ &\;\vdots\\ &= \frac{|a_{k+1}|\cdots|a_n|(||P_n^{k+1}||\cdots||P_n^n||)(||P_{n+m}^{k+1}||\cdots||P_{n+m}^n||)}{(Q_n^{k+1}\cdots Q_n^n)(Q_{n+m}^{k+1}\cdots Q_{n+m}^n)}\Delta_n,\end{aligned}$$

where

$$\begin{aligned}\Delta_n &= ||Q_n^n P_{n+m}^n - P_n^n Q_{n+m}^n||\\ &= ||B_n Q_{n+m}^n + a_{n+1}\bar{P}_{n+m}^{n+1} - B_n Q_{n+m}^n|| = |a_{n+1}|||P_{n+m}^{n+1}||.\end{aligned}$$

Hence, we have

$$\begin{aligned}\Delta_k &= \frac{|a_{k+1}|\cdots|a_{n+1}|(||P_n^{k+1}||\cdots||P_n^n||)(||P_{n+m}^{k+1}||\cdots||P_{n+m}^{n+1}||)}{(Q_n^{k+1}\cdots Q_n^n)(Q_{n+m}^{k+1}\cdots Q_{n+m}^n)}\\ &= |a_{k+1}|\cdots|a_{n+1}|\sqrt{Q_n^k}\sqrt{Q_{n+m}^k}\sqrt{Q_{n+m}^{n+1}}.\end{aligned}$$

Consequently, according to (3.2), we conclude that

$$\left\|\frac{P^k_{n+m}}{Q^k_{n+m}}-\frac{P^k_n}{Q^k_n}\right\|=\frac{|a_{k+1}|\cdots|a_{n+1}|\sqrt{Q^{n+1}_{n+m}}}{\sqrt{Q^k_n}\sqrt{Q^k_{n+m}}}.$$

In particular, for $k=0$, it follows that

$$\left\|\frac{P^0_{n+m}}{Q^0_{n+m}}-\frac{P^0_n}{Q^0_n}\right\|=\frac{|a_1|\cdots|a_{n+1}|\sqrt{Q^{n+1}_{n+m}}}{\sqrt{Q^0_n}\sqrt{Q^0_{n+m}}}$$

and hence equality (3.1) is proved. □

It is easy now to give an expression for the difference of two consecutive convergents of MVCF. In fact, by Theorem 2 with $p=1,2$ and by using (2.4), it follows immediately that

Corollary 1. *The formula*

$$\left\|\frac{P_{n+1}}{Q_{n+1}}-\frac{P_n}{Q_n}\right\|=\left\|\frac{P^0_{n+1}}{Q^0_{n+1}}-\frac{P^0_n}{Q^0_n}\right\|=\frac{|a_1|\cdots|a_{n+1}|}{\sqrt{Q^0_n}\sqrt{Q^0_{n+1}}}\quad (n=0,1,\ldots) \tag{3.3}$$

holds true for two approximants P_{n+1}/Q_{n+1} and P_n/Q_n of matrix-valued continued fraction (2.1), *and the formula*

$$\begin{aligned}\left\|\frac{P_{n+2}}{Q_{n+2}}-\frac{P_n}{Q_n}\right\|&=\left\|\frac{P^0_{n+2}}{Q^0_{n+2}}-\frac{P^0_n}{Q^0_n}\right\|\\&=\frac{|a_1|\cdots|a_{n+1}|\|B_{n+2}\|}{\sqrt{Q^0_n}\sqrt{Q^0_{n+2}}}\quad (n=0,1,\ldots)\end{aligned} \tag{3.4}$$

holds true for two approximants P_{n+2}/Q_{n+2} and P_n/Q_n of matrix-valued continued fraction (2.1).

In the scalar case, namely $k=1$ and $l=1$, B_n $(n=0,1,2,\ldots)$ are scalars. Using the same idea in Theorem 2, we can prove the following theorem.

Theorem 3. *The formula*

$$\frac{P_{n+m}}{Q_{n+m}}-\frac{P_n}{Q_n}=(-1)^n\frac{a_1\cdots a_{n+1}(\bar{P}^1_n\cdots\bar{P}^n_n)(\bar{P}^1_{n+m}\cdots\bar{P}^{n+1}_{n+m})}{(Q^0_n\cdots Q^n_n)(Q^0_{n+m}\cdots Q^n_{n+m})}\quad \forall m,n\in\mathbb{N} \tag{3.5}$$

holds true for any $(m+n)$th approximant P^0_{n+m}/Q^0_{n+m} (or $\frac{P_{n+m}}{Q_{n+m}}$) and nth approximant P^0_n/Q^0_n (or $\frac{P_n}{Q_n}$) of complex continued fraction (1.1), *where P^i_n and Q^i_n are decided by* (2.4)–(2.6).

Further, (3.1) is valid.

Obviously, Formula (3.5) is different from the following formula [see [22]]:

$$\frac{P_{n+m}}{Q_{n+m}}-\frac{P_n}{Q_n}=\frac{-f_n^{(m)}}{h_n+f_n^{(m)}}\left(\frac{P_n}{Q_n}-\frac{P_{n-1}}{Q_{n-1}}\right) \text{ for } n=1,2,\dots,$$

where $f_n^{(m)}=\frac{a_{n+1}|}{|b_{n+1}}+\cdots+\frac{a_{n+m}|}{|b_{n+m}}$, $h_n=-S_n^{-1}(\infty)$ for $n=0,1,2,\dots;\ m=1,2,\dots,n$ and where

$$\begin{cases} S_0(w)=s_0(w),\ S_n(w)=S_{n-1}(s_n(w)) & n=1,2,\dots \\ s_0(w)=b_0+w,\ s_n(w)=\dfrac{a_n}{b_n+w} & n=1,2,\dots. \end{cases}$$

4. Convergence theorems and truncation error

The applications of MVCF (2.1) are often tied to their possible convergence. It is therefore important to have convergence criteria that are easy to check and cover large classes of MVCFs. In this section, we will show that relation (3.1) has wide applications in the convergence of MVCF.

A nonterminating matrix-valued continued fraction (2.1) is called convergent if the sequence $\{R_n\}$ of approximants is convergent, that is

$$\lim_{n\to\infty} R_n=\lim_{n\to\infty}\frac{P_n^0}{Q_n^0}=\lim_{n\to\infty}\frac{P_n}{Q_n}=R$$

and the matrix R is taken as the value of the matrix-valued continued fraction. But if no limit exists, then matrix-valued continued fraction (2.1) is called divergent and no matrix value is assigned to it.

Clearly, according to the Cauchy criterion for convergence, the above statements are equivalent to the following:

The value of the matrix-valued continued fraction exists if conditions (a) and (b) below are satisfied:

(a) At most a finite number of the denominators B_k vanish.
(b) For every positive ε, there exists an N such that, for $n\geqslant N$

$$\left|\left|\frac{P_{n+m}}{Q_{n+m}}-\frac{P_n}{Q_n}\right|\right|<\varepsilon \quad \text{for all positive } m. \tag{4.1}$$

Next, Using Theorem 2, we give a necessary condition for the convergence of MVCF.

Theorem 4. *The matrix-valued continued fraction* (2.1) *with all* $a_n=1$ *diverges if* $\sum_i ||B_i||=M<\infty$.

Proof. First, the following inequalities are easily proved by induction on k,

$$\sqrt{Q_n^k} \leqslant (1+||B_n||)(1+||B_{n-1}||)\cdots(1+||B_{k+1}||) \quad \text{for all } n \in \mathbb{N}, \text{ and } 0 \leqslant k < n. \tag{4.2}$$

In fact, it follows from $Q_n^{n-1} = ||B_n||^2$ in (2.4) that

$$\sqrt{Q_n^{n-1}} \leqslant (1+||B_n||)$$

which proves (4.2) for $k = n-1$.

Next, assume that for all k not exceeding n

$$Q_n^k \leqslant (1+||B_n||)(1+||B_{n-1}||)\cdots(1+||B_{k+1}||)$$

then from relation (2.6), we have

$$\sqrt{Q_n^{k-1}} \leqslant ||B_k||\sqrt{Q_n^k} + \sqrt{Q_n^{k+1}}.$$

Hence, by induction on k, it follows that

$$\begin{aligned}\sqrt{Q_n^{k-1}} &\leqslant ||B_k|| \prod_{i=k+1}^{n} (1+||B_i||) + (1+||B_n||)(1+||B_{n-1}||)\cdots(1+||B_{k+2}||) \\ &\leqslant (1+||B_n||)(1+||B_{n-1}||)\cdots(1+||B_{k+2}||)(||B_k||(1+||B_{k+1}||)+1) \\ &\leqslant (1+||B_n||)(1+||B_{n-1}||)\cdots(1+||B_{k+2}||)(1+||B_{k+1}||)(1+||B_k||)\end{aligned}$$

which proves (4.2).

Moreover, by (4.2) with $k=0$, it follows in particular that

$$\sqrt{Q_n^0} \leqslant (1+||B_n||)(1+||B_{n-1}||)\cdots(1+||B_1||)$$

and

$$\sqrt{Q_{n+1}^0} \leqslant (1+||B_{n+1}||)(1+||B_n||)\cdots(1+||B_1||).$$

Using the inequality

$$1 + x \leqslant e^x \quad \text{for } x > 0$$

we have

$$\sqrt{Q_n^0} \leqslant \exp(||B_n||)\exp(||B_{n-1}||)\cdots\exp(||B_1||) < \exp\left(\sum_{i=1}^{\infty} ||B_i||\right) = e^M \tag{4.3}$$

and

$$\begin{aligned}\sqrt{Q_{n+1}^0} &\leqslant \exp(||B_{n+1}||)\exp(||B_n||)\cdots\exp(||B_1||) \\ &< \exp\left(\sum_{i=1}^{\infty} ||B_i||\right) = e^M.\end{aligned} \tag{4.4}$$

Now, assume that the continued fraction $B_0 + K(1/B_n)$ converges, then from (3.4), it follows that

$$\lim_{n\to\infty} \left\|\frac{P_{n+1}^0}{Q_{n+1}^0} - \frac{P_n^0}{Q_n^0}\right\| = \lim_{n\to\infty} \frac{1}{\sqrt{Q_n^0}\sqrt{Q_{n+1}^0}} = 0.$$

But from (4.3) and (4.4), we get

$$\frac{1}{\sqrt{Q_n^0}\sqrt{Q_{n+1}^0}} > \frac{1}{e^{2M}} > 0, \quad \text{a contradiction.}$$

Hence the continued fraction $B_0 + K(1/B_n)$ diverges. □

Theorem 5. *Let all the elements of B_n $(n = 1, 2, \dots)$ be positive, namely B_n are positive matrices for all n, then the matrix-valued continued fraction $K(1/B_n)$ converges if and only if $\sum_{n=1}^{\infty} ||B_n|| = \infty$.*

Proof. If $\sum_{n=1}^{\infty} ||B_n|| = \infty$, then $K(1/B_n)$ diverges by Theorem 4. Let $\sum_{n=1}^{\infty} ||B_n|| = \infty$. To prove the convergence of $K(1/B_n)$, it suffices to prove that the sequence $\left\{\frac{P_n^0}{Q_n^0}\right\}$ of approximants is a Cauchy sequence, namely, we need to prove, for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \left\|\frac{P_{n+m}^0}{Q_{n+m}^0} - \frac{P_n^0}{Q_n^0}\right\| = 0 \tag{4.5}$$

is true.

Since the B_n $(n = 0, 1, \dots)$ are positive matrices, it follows from (2.5) that also P_n^k are positive matrices for all $n \in \mathbb{N}$, and $0 \leqslant k < n$. Hence, it follows from (2.4) and (2.6) that

$$Q_{2n}^k \geqslant ||B_{k+1}||^2 Q_{2n}^{k+1} + Q_{2n}^{k+2} \quad \text{with } Q_{2n}^{2n} = 1, \; Q_{2n}^{2n-1} = ||B_{2n}||^2.$$

Clearly, the above inequalities imply that

$$Q_{2n}^0 > Q_{2n}^2 > \cdots > Q_{2n}^{2n} = 1$$

and

$$Q_{2n}^1 > Q_{2n}^3 > \cdots > Q_{2n}^{2n-1} = ||B_{2n}||^2.$$

On the other hand, if we let $B_i(s,t)$, $P_n^{i+1}(s,t)$ denote the elements from the sth row and tth column of the matrix B_i and P_n^{i+1}, respectively, it follows from (2.5) that

$$P_n^i(s,t) = B_i(s,t) Q_n^i + P_n^{i+1}(s,t) \geqslant B_i(s,t) Q_n^i + P_n^{i+2}(s,t).$$

In particular, we have

$$P_{2n}^{2i}(s,t) \geqslant B_{2i}(s,t) Q_{2n}^{2i} + P_{2n}^{2i+2}(s,t)$$

and it follows from $Q_{2n}^{2i} > 1$ that

$$P_{2n}^{2i}(s,t) > B_{2i}(s,t) + P_{2n}^{2i+2}(s,t) \quad (i = 0, 1, \dots, n-1).$$

Expanding $P_{2n}^{2i}(s,t)$ from $i=0$ to $i=n-1$ in the same manner, and continuing the process, one obtains

$$P_{2n}^{0}(s,t) > B_0(s,t) + P_{2n}^{2}(s,t) > B_0(s,t) + B_2(s,t) + P_{2n}^{4}(s,t) > \cdots > B_0(s,t) + B_2(s,t) + \cdots + B_{2n}(s,t). \quad (4.6)$$

In the same way, one can prove that

$$\begin{cases} Q_{2n+1}^{2n+1} = 1, \;\; Q_{2n+1}^{2n} = \|B_{2n+1}\|^2, \;\; Q_{2n+1}^{0} > Q_{2n+1}^{2} > \cdots > Q_{2n+1}^{2n} = \|B_{2n+1}\|^2, \\ \quad Q_{2n+1}^{1} > Q_{2n+1}^{3} > \cdots > Q_{2n+1}^{2n+1} = 1 \end{cases}$$

furthermore

$$P_{2n+1}^{1}(s,t) > B_1(s,t) + B_3(s,t) + \cdots + B_{2n+1}(s,t). \quad (4.7)$$

Since $\sum_{n=1}^{\infty} \|B_n\| = \infty$, then using a proof by contradiction, at least there exists some s_0, t_0 $(1 \leqslant s_0 \leqslant k, 1 \leqslant t_0 \leqslant l)$ such that $\sum_{n=1}^{\infty} |B_n(s_0,t_0)| = \infty$.

From (4.6) and (4.7), we have

$$\lim_{n\to\infty} P_{2n}^{0}(s_0,t_0) = \infty \quad (4.8)$$

or

$$\lim_{n\to\infty} P_{2n+1}^{1}(s_0,t_0) = \infty. \quad (4.9)$$

Without lose of generality, we assume that $\lim_{n\to\infty} P_{2n}^{0}(s_0,t_0) = \infty$. From this, it follows that

$$\lim_{n\to\infty} \|P_{2n}^{0}\| = \infty. \quad (4.10)$$

According to (2.6), it follows that

$$Q_{2n}^{0} \geqslant \|B_1\|^2 Q_{2n}^{1}.$$

From this and (2.8), one obtains

$$\|P_{2n}^{0}\|^2 = Q_{2n}^{0} Q_{2n}^{1} \leqslant \frac{(Q_{2n}^{0})^2}{\|B_1\|^2}$$

so that

$$Q_{2n}^{0} \geqslant \|B_1\| \|P_{2n}^{0}\|.$$

On the base of (4.10), we have

$$\lim_{n\to\infty} Q_{2n}^{0} = \infty$$

and taking into account inequalities (4.6) and formula (3.1), we find

$$\left\| \frac{P_{2n+m}^{0}}{Q_{2n+m}^{0}} - \frac{P_{2n}^{0}}{Q_{2n}^{0}} \right\| = \frac{\sqrt{Q_{2n+m}^{2n+1}}}{\sqrt{Q_{2n}^{0}}\sqrt{Q_{2n+m}^{0}}} < \frac{1}{\|B_1\|\sqrt{Q_{2n}^{0}}} \to 0 \quad (n\to\infty) \quad (4.11)$$

and

$$\begin{aligned}\left\|\frac{P^0_{2n+m}}{Q^0_{2n+m}}-\frac{P^0_{2n-1}}{Q^0_{2n-1}}\right\| &\leqslant \left\|\frac{P^0_{2n+m}}{Q^0_{2n+m}}-\frac{P^0_{2n}}{Q^0_{2n}}\right\|+\left\|\frac{P^0_{2n-1}}{Q^0_{2n-1}}-\frac{P^0_{2n}}{Q^0_{2n}}\right\| \\ &< \frac{\sqrt{Q^{2n+1}_{2n+m}}}{\sqrt{Q^0_{2n}}\sqrt{Q^0_{2n+m}}}+\frac{1}{\sqrt{Q^0_{2n-1}}\sqrt{Q^0_{2n}}} \\ &\leqslant \frac{2}{||B_1||\sqrt{Q^0_{2n}}}\to 0 \quad (n\to\infty). \end{aligned} \tag{4.12}$$

Hence, from (4.11) and (4.12), we complete the proof of (4.5), namely, the sequence $\{\frac{P^0_n}{Q^0_n}\}$ of approximants is a Cauchy sequence and thus according to the Cauchy criterion for convergence, it converges. □

Obviously, Theorems 4 and 5 are exact generalizations of the scalar results [see [22].

Theorem 6. *If the inequalities*

$$||B_i||\geqslant d+|a_{i+1}| \quad \textit{for all } i=0,1,2,\ldots, \textit{ where } d>1 \tag{4.13}$$

hold true, then matrix-valued continued fraction (2.1) *converges to matrix value R and the truncation error* $|R-R_n|\leqslant(\frac{1}{d})^n$.

Proof. First, we can prove the following inequality by induction on k.

$$Q^k_n\geqslant Q^{k+1}_n \quad \text{for } \forall n \text{ and } k=0,1,\ldots,n-1, \tag{4.14}$$

when $k=n-1$, inequality (4.14) is immediate from relation (2.4).

Now, let formula (4.14) be true for all the natural number $k+1$ not exceeding n, then from relation (2.6) and by the induction hypothesis, we obtain

$$\sqrt{Q^k_n}\geqslant ||B_{k+1}||\sqrt{Q^{k+1}_n}-|a_{k+2}|\sqrt{Q^{k+2}_n}, \tag{4.15}$$

$$\begin{aligned}\sqrt{Q^k_n}-\sqrt{Q^{k+1}_n} &\geqslant |a_{k+2}|\left(\sqrt{Q^{k+1}_n}-\sqrt{Q^{k+2}_n}\right) \\ &\geqslant |a_{k+2}||a_{k+3}|\cdots|a_n|\left(\sqrt{Q^{n-1}_n}-\sqrt{Q^n_n}\right) \\ &\geqslant |a_{k+2}|\cdots|a_n|(||B_n||-1)\geqslant 0 \end{aligned} \tag{4.16}$$

which completes the proof of (4.14).

In particular, from (4.16) and (4.13), we get

$$\sqrt{Q^0_n}\geqslant |a_1||a_2|\cdots|a_n|(||B_n||-1)\geqslant |a_1||a_2|\cdots|a_{n+1}|. \tag{4.17}$$

Next, from relations (4.13)–(4.15), for $\forall m, n > 0$ and $0 \leqslant k < n$, we have

$$\begin{aligned}\sqrt{Q^{k}_{n+m}} &\geqslant ||B_{k+1}||\sqrt{Q^{k+1}_{n+m}} - |a_{k+2}|\sqrt{Q^{k+2}_{n+m}}\\ &\geqslant (||B_{k+1}|| - |a_{k+2}|)\sqrt{Q^{k+1}_{n+m}}\\ &\geqslant d\sqrt{Q^{k+1}_{n+m}}. \end{aligned} \tag{4.18}$$

Continuing the above process, we obtain

$$\sqrt{Q^{0}_{n+m}} \geqslant d\sqrt{Q^{1}_{n+m}} \geqslant \cdots \geqslant d^{n+1}\sqrt{Q^{n+1}_{n+m}}. \tag{4.19}$$

Thus, taking into account inequality (4.17), (4.19) and formula (3.1), it follows for $\forall m > 0$, that

$$\left|\left|\frac{p^0_{n+m}}{Q^0_{n+m}} - \frac{p^0_n}{Q^0_n}\right|\right| = \frac{|a_1||a_2|\cdots|a_{n+1}|\sqrt{Q^{n+1}_{n+m}}}{\sqrt{Q^0_n}\sqrt{Q^0_{n+m}}} < \frac{1}{d^{n+1}} \to 0 \quad (n \to \infty). \tag{4.20}$$

Hence, from (4.20), the sequence $\{\frac{P^0_n}{Q^0_n}\}$ of approximants is a Cauchy sequence, and thus according to the Cauchy criterion for convergence, it converges to a matrix value R. From (4.20), clearly, the truncation error bounds $|R - R_n| \leqslant (\frac{1}{d})^{n+1}$, which completes the proof of Theorem 6. □

It would be desirable to extend the theorem to the case $d = 1$ so as to get a Śleszyński–Pringsheim-like theorem, But it appears that it will require a proof of a different type than that above. Here, we need to point out Theorem 6 is a bit different from the Śleszyński–Pringsheim Theorem. The Śleszyński–Pringsheim Theorem for continued fractions

$$K(a_n/b_n) = \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n} + \cdots, \tag{4.21}$$

where $a_n, b_n \in \mathbb{C}$ with all $a_n \neq 0$, says that $K(a_n/b_n)$ converges to a value f if

$$|b_n| \geqslant 1 + |a_n| \quad \text{for all } n. \tag{4.22}$$

But in our case, the condition of theorem is $|b_n| \geqslant d + |a_{n+1}|,\ d > 1$. In addition, as we know, by means of (4.22) and equivalence transformation of CF, it can give new convergence criteria, but it appears improbable to get Theorem 6 by this method. Therefore, we can say that Theorem 6 appears a new convergence criteria for continued fraction (4.21).

Proceeding with the similar method in Theorems 5 and 6, we can prove the following conclusions. Here, we only give a sketch of the proofs of these theorems, we leave the details to the reader.

Theorem 7 (Worpitzky-like theorem). *Let $||B_n|| = 2,\ |a_n| \leqslant 1$ for all n, then MCVF $K(\frac{1}{B_n})$ converges, and if $\lim_{n\to\infty} R_n = R$, one gets*

$$||R_n - R|| \leqslant \frac{|a_1 a_2 \cdots a_n|}{2 + \sum_{i=2}^{n} |a_i \cdots a_n|}.$$

Moreover, if $\sum_k |a_1 \cdots a_k| = a$, then $||R_n - R|| \leqslant \frac{a}{n^2}$.

Proof. As we saw previously in the proof of Theorem 6, paying attention to $||B_n|| = 2$ in (4.16), we have

$$\sqrt{Q_n^k} \geqslant \sqrt{Q_n^{k+1}} + |a_{k+2}| \cdots |a_n|. \tag{4.23}$$

Repeating the above proceeding from $k = 0$ to $n - 1$, we get

$$\sqrt{Q_n^0} \geqslant \sqrt{Q_n^{n-1}} + \sum_{k=0}^{n-2} |a_{k+2}| \cdots |a_n| = 2 + \sum_{k=2}^{n} |a_k| \cdots |a_n|. \tag{4.24}$$

From (4.23) we immediately obtain

$$\sqrt{Q_{n+m}^0} \geqslant \sqrt{Q_{n+m}^{n+1}}. \tag{4.25}$$

By using Theorem 2 and from (4.24) and (4.25), Theorem 7 is valid.

In the scalar case, using a geometric argument, the similar truncation error results are stated in [4]. □

Theorem 8. *If the inequalities*

$$||B_k|| \geqslant 2 \quad \textit{for all } k = 1, 2, \ldots$$

hold true, then matrix-valued continued fraction $\sum_{i=1}^{\infty} \frac{1|}{|B_n}$ converges to matrix value R and the truncation error $|R - R_n| = 0(\frac{1}{n})$.

Proof. Paying attention to $||B_n|| \geqslant 2$ and $|a_n| = 1$ for all n in (4.16), we get

$$\sqrt{Q_n^0} \geqslant n + 1. \tag{4.26}$$

Using (4.18), we have

$$\sqrt{Q_{n+m}^0} \geqslant \sqrt{Q_{n+m}^{n+1}}. \tag{4.27}$$

From (4.26), (4.27) and Theorem 2, we find Theorem 8 is true. □

Theorem 9. *If B_k are positive matrices and a_k $(k = 0, 1, 2 \ldots)$ are positive, and for all k, the inequalities*

$$||B_k|| \geqslant a_k, \quad ||B_k|| \geqslant d, \quad \textit{where } d > 0$$

are satisfied, then matrix-valued continued fraction (2.1) *converges.*

Proof. Since B_k are positive matrices and a_k $(k = 0, 1, 2 \dots)$ are positive, using (2.6), It is easy to see that

$$\begin{cases} Q_n^k \geqslant ||B_{k+1}||^2 Q_n^{k+1}, \\ Q_n^k \geqslant |a_{k+2}|^2 Q_n^{k+2}. \end{cases} \tag{4.28}$$

Furthermore, using $||B_k|| \geqslant a_k$, $||B_k|| \geqslant d$, it follows from (2.6) and (4.28) that

$$Q_n^k \geqslant |a_{k+2}|^2 (1 + d^2) Q_n^{k+2}. \tag{4.29}$$

From (4.28) and (4.29), we see that

$$\begin{cases} Q_{2n}^0 \geqslant |a_2|^2 (1 + d^2) Q_{2n}^2 \geqslant \cdots \geqslant (1 + d^2)^n \prod_{k=1}^n |a_{2k}|^2, \\ Q_{2n+m}^0 \geqslant ||B_1||^2 Q_{2n+m}^1 \geqslant \cdots \geqslant ||B_1||^2 (1 + d^2)^n Q_{2n+m}^{2n+1} \prod_{k=1}^n |a_{2k+1}|^2. \end{cases} \tag{4.30}$$

From (4.30), we have

$$Q_{2n}^0 Q_{2n+m}^0 \geqslant ||B_1||^2 (1 + d^2)^{2n} Q_{2n+m}^{2n+1} \prod_{k=2}^{2n+1} |a_k|. \tag{4.31}$$

Similarly, we obtain

$$Q_{2n+1}^0 Q_{2n+1+m}^0 \geqslant ||B_1||^2 (1 + d^2)^{2n+1} Q_{2n+m}^{2n+2} \prod_{k=2}^{2n+2} |a_k|. \tag{4.32}$$

From (4.31) and (4.32), using Theorem 2, the proof of Theorem 9 is straightforward and so is omitted. □

It is clear that Theorems 7, 8 and 9 are exact generalizations of the scalar results [see [4,6]].

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